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Abstract: It is known that rigid body models with two controls cannot be locally asymptotically stabilized by smooth feedbacks which are functions of the state only. This impossibility does no longer hold when the feedback is also a function of time, or when it is non smooth. A locally stabilizing smooth time-varying feedback is here explicitly derived by using Center Manifold Theory combined with averaging and Lyapunov techniques.

Key-words: Time-varying control, smooth feedback, attitude stabilization.

(Résumé : tsvp)

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Stabilisation par Retour d'Etat Instationnaire d'un Satellite avec Deux Commandes

Résumé : Il est connu que l'orientation d'un corps rigide auquel sont appliquées deux commandes ne peut être stabilisée par retour d'état continu fonction seulement de l'état. La stabilisation par retour d'état continu dépendant également du temps ou par retour d'état discontinu reste cependant possible. Dans ce rapport, un retour d' état instationnaire infiniment différentiable et localement asymptotiquement stabilisant est synthétisé en utilisant la Théorie de la Variété Centre et des techniques de Lyapunov.

Mots-clé : Commande instationnaire, retour d' état régulier, stabilisation d'attitude.

1 Introduction

The attitude control of a rigid spacecraft operating in degraded mode, i.e with only one or two controls, has already been much studied in the literature.

This type of system has for example been used to illustrate several aspects of nonlinear controllability. Let us just mention the results of Bonnard [3] and Crouch [9] proving that the system is globally controllable except for some exceptional locations of the actuators, and the work by Kerai [12] which establishes small time local controllability (STLC) in the same situations with two controls.

The present paper focuses on the attitude stabilization problem. Some early results may be found in [9] where controllability results are used to transform open-loop control strategies into a feedback. The feedback so obtained is of course discontinuous and a simplified alternative method is proposed in [13].

We are concerned here with continuous feedback. The related but simpler problem consisting of stabilizing the angular velocity of the spacecraft with one or two actuators has been investigated by several authors, see Aeyels [1], Sontag and Sussmann [21], and more recently Byrnes and Isidori [2]. In the past, it seems that smooth stabilization of the attitude had been implicitly ruled out because this system belongs to the class of systems, singled out by Brockett [4], which are controllable but cannot be stabilized via continuous state feedback. For instance, it is stated in Byrnes and Isidori [2] that the model of a spacecraft with two actuators does not satisfy Brockett's necessary condition for smooth feedback stabilizability and hence that there exists no continuous state feedback law which locally asymptotically stabilizes the rigid spacecraft about a reference nominal frame.

On the other hand, an article by Samson [18] has recently triggered the discovery that many systems of interest which cannot be stabilized by continuous state-feedback can in fact be stabilized by smooth "*time-varying*" feedback. Research on time-varying control has then expanded quickly. In particular Coron (see e.g [6], [7]) established that, under very mild assumptions, any STLC system is stabilizable (in finite time) by continuous time-varying feedback. Several constructive approaches have also been developed for driftless systems which do not respect Brockett's necessary condition (see e.g [16], [8], [17]). However their extensions to systems with drift is not straightforward. In [14], Samson and Morin present a first attempt to stabilize the spacecraft with time-varying controls. No stability proof is provided there, but the given control laws display stabilization properties on simulations.

Existence of a stabilizing continuous time-varying feedback for a rigid spacecraft with two pairs of gas jets (i.e two controls), except in the cases known to be uncontrollable, follows from the aforementioned result by Kerai on STLC [12] and from Coron's results (see [7, Thm 2.7, Rmk 2.6, note pg 3]).

In the present paper, an explicit smooth time-varying stabilizing feedback is derived [1]. The techniques used for the construction of this control combine center manifold theory with time-averaging and Lyapunov techniques. The idea of applying center manifold theory to time varying stabilization has previously been used in [20], and subsequently in [22], for the stabilization of some driftless systems.

The paper is organized as follows. In Section 2, we recall the equations of a rigid body using a parameterization of rotations derived from quaternions and precisely state the control objective. A simplified control problem is treated in Section 3 and our main result is given in Section 4.

Throughout the paper, we will use the following notations:

- $|\cdot|$ denotes the Euclidean norm.
- $O_t^q(X)$ stands for any continuous function $f : \mathbb{R}^n \times \mathbb{R} \longmapsto \mathbb{R}^p$ such that:

$$\exists \delta > 0,\ \exists K \ :\ |X| \leq \delta \Longrightarrow |f(X,t)| \leq K|X|^q \ .$$

2 Problem Statement

Let us consider:

- a frame F_0 attached to the spacecraft and whose axes correspond to the principal inertia axes of the body.
- a fixed frame F_1 whose attitude is the desired one for F_0.

and denote:

- ω: the angular velocity vector of the frame F_0 with respect to the frame F_1, expressed in the basis of F_0.
- J: the (diagonal) inertia matrix:

$$J = \begin{pmatrix} j_1 & 0 & 0 \\ 0 & j_2 & 0 \\ 0 & 0 & j_3 \end{pmatrix} \tag{1}$$

[1] While preparing this manuscript, we have been informed of an independent work by R. Montgomery, S.S. Sastry, and G.C. Walsh in the same direction. No preprint has yet been communicated to us.

- $S(\omega)$: the matrix representation of the cross product:

$$S(\omega) = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \tag{2}$$

If R is the rotation matrix representing the attitude of F_1 with respect to F_0 (and whose columns vectors are the basis vectors of F_1 expressed in F_0) , we get the well known equations:

$$\left\{ \begin{array}{lcl} \dot{R} & = & S(\omega)R \\ J\dot{\omega} & = & S(\omega)J\omega + (\tau_1, \tau_2, 0)^T \end{array} \right. \tag{3}$$

where the τ_i are the torques applied to the rigid body.

This is a control system with two scalar inputs τ_1 and τ_2 and state space $SO(3) \times \mathbb{R}^3$. Our objective is to find a control $(\tau_1(t, R, \omega), \tau_2(t, R, \omega))$ periodic with respect to time, which locally asymptotically stabilizes the point $(I_3, 0)$ of $SO(3) \times \mathbb{R}^3$.

In order to control the body rotations, a preliminary step traditionally consists in defining a minimal set of local coordinates for the parametrization of $SO(3)$ around I_3. A common choice is the Euler angles. We make a different choice that has the advantage of yielding polynomial equations. Let S^3 be the three-dimensional sphere, identified to the multiplicative group of unitary quaternions $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ with $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$. The classical notation $r(\lambda) = \lambda_0$, $p(\lambda) = (\lambda_1, \lambda_2, \lambda_3)$ is used to distinguish the real part of the quaternion from the pure imaginary part. In [19] for example, one may find the description of a mapping φ from S^3 to $SO(3)$ which is a Lie group homomorphism and a twofold covering of $SO(3)$, i.e. φ is a local diffeomorphism and each element of $SO(3)$ is the image of exactly two unitary quaternions (λ and $-\lambda$). It is also established in [19] that (3) lifts to the following control system on $S^3 \times \mathbb{R}^3$:

$$\left\{ \begin{array}{lcl} \dot{\lambda} & = & \frac{1}{2} \begin{pmatrix} 0 & -\omega^T \\ \omega & S(\omega) \end{pmatrix} \lambda \\ J\dot{\omega} & = & S(\omega)J\omega + (\tau_1, \tau_2, 0)^T \end{array} \right. \tag{4}$$

The restriction of φ to the hemisphere $\{r(\lambda) > 0\}$ is a diffeomorphism from this hemisphere to the open subset of $SO(3)$ consisting of rotations with angle different from π. Its inverse maps a rotation of angle $\theta \in]-\pi, \pi[$ around the axis defined by the unit vector $\vec{u}$ to the quaternion ($r(\lambda) = \cos\frac{\theta}{2}$, $p(\lambda) = \sin\frac{\theta}{2}\vec{u}$). We choose, on this hemisphere, the following coordinates

$$X \;=\; (x_1, x_2, x_3) \;=\; \frac{p(\lambda)}{r(\lambda)} \;=\; (\frac{\lambda_1}{\lambda_0}, \frac{\lambda_2}{\lambda_0}, \frac{\lambda_3}{\lambda_0})$$

The mapping $\lambda \mapsto X$ is a diffeomorphism from this hemisphere to $\mathbb{R}^3$, and X is a minimal parameterization of the rotations of angle different from π. It is simple to write the system (4) –and hence the original system– in the coordinates (X, ω) . One obtains:

$$\left\{ \begin{array}{lcl} \dot{X} & = & \frac{1}{2}(\ \omega \ + \ S(\omega)\, X \ + \ \langle \omega, X \rangle\, X\ ) \\ \dot{\omega}_1 & = & c_1\, \omega_2 \omega_3 + u_1 \\ \dot{\omega}_2 & = & c_2\, \omega_1 \omega_3 + u_2 \\ \dot{\omega}_3 & = & c_3\, \omega_1 \omega_2 \end{array} \right. \tag{5}$$

with $c_1 = \dfrac{j_2 - j_3}{j_1}, c_2 = \dfrac{j_3 - j_1}{j_2}, c_3 = \dfrac{j_1 - j_2}{j_3}, u_1 = \dfrac{\tau_1}{j_1}$, and $u_2 = \dfrac{\tau_2}{j_2}$.

It is of course assumed that $c_3 \neq 0$, since otherwise the system would not be controllable nor stabilizable. Moreover we may also assume that $c_3 > 0$, due to the fact that the change of variables

$$(x_1, x_2, x_3, \omega_1, \omega_2, \omega_3, u_1, u_2) \longmapsto (x_2, x_1, -x_3, \omega_2, \omega_1, -\omega_3, u_2, u_1)$$

leaves the equation (5) unchanged, except for the parameters (c_1, c_2, c_3) which are changed into $(-c_2, -c_1, -c_3)$.

The problem addressed in this paper consists of finding a smooth feedback control law which asymptotically stabilizes the origin of (5). The problem is first simplified by considering the following reduced order system obtained by taking $v_1 \stackrel{\Delta}{=} \omega_1$ and $v_2 \stackrel{\Delta}{=} \omega_2$ as control variables :

$$\begin{array}{lcl} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} & = & \frac{1}{2}[\begin{pmatrix} v_1 \\ v_2 \\ \omega_3 \end{pmatrix} + \begin{pmatrix} 0 & \omega_3 & -v_2 \\ -\omega_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + (v_1 x_1 + v_2 x_2 + \omega_3 x_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}] \\ \dot{\omega}_3 & = & c_3\, v_1\, v_2 \end{array} \tag{6}$$

This reduced system potentially contains the difficulties associated with the original system.

A solution to the problem of stabilizing the origin of the reduced system (6) is described in the next section. This solution is then adapted, in Section 4, to the original system (5).

3 Stabilization of the reduced system with velocity controls

A possible control law and the corresponding stabilization result are stated next:

Theorem 1 *The smooth time-varying control:*

$$\begin{cases} v_1(X,\omega_3,t) &=& 2g_1\dot{h}_1 + h_1\dfrac{\partial g_1}{\partial x_3}\omega_3 - 2k_1(x_1 - g_1h_1) \quad k_1 > 0 \\ v_2(X,\omega_3,t) &=& 2g_2\dot{h}_2 - x_3v_1(X,\omega_3,t) + x_1\omega_3 \\ && +h_2\dfrac{\partial g_2}{\partial x_3}\omega_3 - 2k_2(x_2 - g_2h_2) \qquad k_2 > 0 \end{cases} \tag{7}$$

with

$$\begin{cases} g_1 &=& \alpha x_3 + \beta\omega_3 \\ g_2 &=& x_3^2 + \omega_3^2 \\ h_1 &=& a_1 \sin t \\ h_2 &=& a_2 \sin t + a_3 \cos t \end{cases} \tag{8}$$

and α, β, a_1, a_2 and a_3 being real numbers such that:

$$a_1 > 0, \quad a_2 < 0, \quad a_3 > 0, \quad \alpha = -\frac{a_3^2}{8a_1a_2}, \quad \beta = \frac{a_3}{4a_1} \tag{9}$$

locally asymptotically stabilizes the origin of (6).

The proof developed in the sequel of the section consists in a step by step construction of the control law (7).

From the first two equations of system (6), it is clearly possible to create for the closed loop system, via an adequate choice of v_1 and v_2, a two-dimensional attractive center manifold. We choose a center manifold which is time varying (in order to circumvent Brockett's necessary condition) and arbitrary assign an approximation given by $x_1 = g_1(x_3,\omega_3)h_1(t)$ and $x_2 = g_2(x_3,\omega_3)h_2(t)$ where h_1 and h_2 are periodic functions of time.

Accordingly, and in order to simplify the exposition, we make the following change of variables:

$$\begin{cases} z_1 &=& x_1 - g_1(x_3,\omega_3)\,h_1(t) \\ z_2 &=& x_2 - g_2(x_3,\omega_3)\,h_2(t) \end{cases} \tag{10}$$

The control (7) of Theorem 1 can then be rewritten in the new coordinates $(z_1, z_2, x_3, \omega_3)$ as:

$$\begin{cases} v_1(z_1,x_3,\omega_3,t) &=& 2g_1\dot{h}_1 + h_1\frac{\partial g_1}{\partial x_3}\omega_3 - 2k_1z_1, & k_1 > 0 \\ v_2(z_1,z_2,x_3,\omega_3,t) &=& 2g_2\dot{h}_2 - x_3v_1 + (z_1 + g_1h_1)\omega_3 + h_2\frac{\partial g_2}{\partial x_3}\omega_3 - 2k_2z_2, & k_2 > 0 \end{cases} \tag{11}$$

Lemma 1 *With the controls v_1 and v_2 given by (11), where g_1 and g_2 are homogeneous polynomials in x_3 and ω_3 of degree 1 and 2 respectively, and h_1 and h_2 smooth periodic functions of time, the closed-loop system may be written, in coordinates $(z_1, z_2, x_3, \omega_3)$, as:*

$$\left\{\begin{array}{ccl} \dot{z}_1 & = & -k_1 z_1 + l_1(z_1, z_2, x_3, \omega_3, t) \\ \dot{z}_2 & = & -k_2 z_2 + l_2(z_1, z_2, x_3, \omega_3, t) \\ \begin{pmatrix} \dot{x}_3 \\ \dot{\omega}_3 \end{pmatrix} & = & \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ \omega_3 \end{pmatrix} + f(z_1, z_2, x_3, \omega_3, t) \end{array}\right. \tag{12}$$

where $f = (f_1, f_2)^T$ and $l = (l_1, l_2)^T$ are:

- *smooth functions, periodic with respect to time, which vanish at $(0, 0, 0, 0, t)$ and whose first order derivatives also vanish at these points.*
- *such that:*

$$\begin{array}{rcl} f_1(z_1, z_2, x_3, \omega_3, t) & = & -\frac{1}{2}(z_2 + h_2(t)g_2)v_1 + \frac{1}{2}(z_1 + h_1(t)g_1)v_2 \\ & & + \frac{1}{2}\left[(z_1 + h_1(t)g_1)v_1 + (z_2 + h_2(t)g_2)v_2 + x_3\omega_3\right] x_3 \\ f_2(z_1, z_2, x_3, \omega_3, t) & = & c_3 v_1 v_2 \end{array} \tag{13}$$

$$l_1(0, 0, x_3, \omega_3, t) = O_t^3(x_3, \omega_3) \tag{14}$$

$$l_2(O_t^3(x_3, \omega_3), 0, x_3, \omega_3, t) = O_t^4(x_3, \omega_3) \tag{15}$$

(Proof in the Appendix)

The closed-loop system (12) is now in the right form for the application of center manifold results. We will specifically use the following lemma, which is not completely standard in that it deals with time-varying periodic vector fields and also involves various orders of approximation of the center manifold.

Lemma 2 *Consider the system*

$$\left\{\begin{array}{ccl} \dot{z} & = & Bz + l(z, x, t) \\ \dot{x} & = & Ax + f(z, x, t) \end{array}\right. \tag{16}$$

with $z \in \mathbb{R}^n, x \in \mathbb{R}^m$, B an upper triangular stable matrix ($b_{ij} = 0$ if $i > j$, $b_{ii} < 0$ for $i = 1, \ldots, n$), A a matrix with eigenvalues having zero real parts, l and f T-periodic functions of class C^2 vanishing at $(0, 0, t)$ and whose first order derivatives also vanish at this point.

Assume that there exists an ordered set of integers q_i*:*

$$2 \leq q_1 \leq q_2 \leq ... \leq q_n \tag{17}$$

such that:

$$\begin{cases} l_1(0,x,t) = O_t^{q_1}(x) \\ \vdots \\ l_i(O_t^{q_1}(x),...,O_t^{q_{i-1}}(x),0,...,0,x,t) = O_t^{q_i}(x) \\ \vdots \\ l_n(O_t^{q_1}(x),...,O_t^{q_{n-1}}(x),0,x,t) = O_t^{q_n}(x) \end{cases} \tag{18}$$

Assume further that the origin of the time-varying system:

$$\dot{x} = Ax + f(\pi_1(x,t),...,\pi_n(x,t),x,t) \tag{19}$$

is locally asymptotically stable whenever $\pi_i(x,t)$ *is any* $O_t^{q_i}(x)$ *function (i=1, ... n). Then, the origin of (16) is locally asymptotically stable.*

(Proof in the Appendix)

In order to prove Theorem 1, one only has to show that this lemma applies, with $q_1 = 3$ and $q_2 = 4$, to the system (12) of lemma 1 for which:

$$n = 2,\ A = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} -k_1 & 0 \\ 0 & -k_2 \end{pmatrix}.$$

To this purpose, it is sufficient to show that the origin of the following system:

$$\begin{pmatrix} \dot{x}_3 \\ \dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ \omega_3 \end{pmatrix} + f(O_t^3(x_3,\omega_3), O_t^4(x_3,\omega_3), x_3, \omega_3, t) \tag{20}$$

is locally asymptotically stable for any $O_t^3(x_3,\omega_3)$ and $O_t^4(x_3,\omega_3)$ functions. Let us hereafter concentrate on the solutions of the system (20).

Using the fact that, from(11),

$$\begin{cases} v_1(O_t^3(x_3,\omega_3),x_3,\omega_3,t) & = & \tilde{v}_1(x_3,\omega_3,t) + O_t^3(x_3,\omega_3) \\ v_2(O_t^3(x_3,\omega_3),O_t^4(x_3,\omega_3),x_3,\omega_3,t) & = & \tilde{v}_2(x_3,\omega_3,t) + O_t^4(x_3,\omega_3) \end{cases} \tag{21}$$

with

$$\begin{cases} \tilde{v}_1(x_3,\omega_3,t) & = & 2g_1\dot{h}_1 + h_1\frac{\partial g_1}{\partial x_3}\omega_3 \\ \tilde{v}_2(x_3,\omega_3,t) & = & 2g_2\dot{h}_2 - x_3\tilde{v}_1 + g_1h_1\omega_3 + h_2\frac{\partial g_2}{\partial x_3}\omega_3\ , \end{cases} \tag{22}$$

we may, from (13), rewrite (20) as :

$$\left\{\begin{array}{lcl}\dot{x}_3 & = & \frac{1}{2}(\omega_3 - g_2h_2\tilde{v}_1 + g_1h_1\tilde{v}_2 + x_3^2\omega_3 + x_3g_1h_1\tilde{v}_1) + O_t^5(x_3,\omega_3) \\ \dot{\omega}_3 & = & c_3\tilde{v}_1\tilde{v}_2 + O_t^5(x_3,\omega_3)\ .\end{array}\right. \tag{23}$$

In order to facilitate the study of this time-varying differential equation, it is possible to introduce a time-varying change of coordinates, the effect of which is to render all terms, up to the fourth order, time-invariant. This possibility is precisely stated in the following lemma which is reminiscent of classical results, reported for example in [10], and similar results used in [22].

Lemma 3 *(Averaging lemma)*
Let A be a $n \times n$ strictly upper triangular matrix ($j \leq i \Rightarrow a_{ij} = 0$), $D(t)$ a $n \times p$ matrix whose components $d_{ij}(t)$ are T-periodic functions of class $C^r (r \geq 0)$, and $\sigma(x)$ a vector-valued function whose components $\sigma_i(x_1, \ldots, x_n)$ are monomials of degree $k \geq 2$.
Then there exists a neighborhood Ω of 0 in $\mathbb{R}^n$ and a C^{r+1} local change of coordinates :

$$(x,t) \mapsto (y,t)$$

defined on $\Omega \times \mathbb{R}$ which maps $(0,t)$ into itself and transforms a system:

$$\dot{x} = Ax + D(t)\sigma(x) + O_t^{2k-1}(x) \tag{24}$$

into a system:

$$\dot{y} = Ay + \overline{D}\sigma(y) + O_t^{2k-1}(y) \tag{25}$$

where $\overline{D}$ is the time average of $D(t)$: $\overline{D} = \frac{1}{T}\int_0^T D(t)\,dt$.

(Proof in the Appendix)

This lemma is here applied, with $k = 3$, to the system (23) for which $n = 2$ and $x = (x_3, \omega_3)$. The right hand side of (23) contains linear terms, terms of order at least equal to five, and a sum of products of time-periodic coefficients with monomials of degree three in the variables x_3 and ω_3. Hence, when writting the system (23) in the form (24), one may take $\sigma(x_3, \omega_3) = (x_3^3, x_3^2\omega_3, x_3\omega_3^2, \omega_3^3)^T$ and determine the time-varying matrix $D(t)$ accordingly.

In view of this lemma, it is equivalent to study the stability of the system (23) and the stability of the corresponding "averaged" system:

$$\left\{\begin{array}{lcl} \dot{x}_3 & = & \frac{1}{2}\omega_3 - 2g_1g_2\overline{\dot{h}_1h_2} + \omega_3 O_t^2(x_3,\omega_3) + O_t^5(x_3,\omega_3) \\ \dot{\omega}_3 & = & c_3[-x_3(4g_1^2\overline{(\dot{h}_1)^2} + \overline{h_1^2}(\frac{\partial g_1}{\partial x_3}\omega_3)^2 \\ & & +2\omega_3\overline{\dot{h}_1h_2}(g_1\frac{\partial g_2}{\partial x_3} - g_2\frac{\partial g_1}{\partial x_3}) \\ & & +\omega_3^2(\overline{h_1h_2}\frac{\partial g_1}{\partial x_3}\frac{\partial g_2}{\partial x_3} + \overline{h_1^2}g_1\frac{\partial g_1}{\partial x_3}) \\ & & +4g_1g_2\overline{\dot{h}_1\dot{h}_2}] + O_t^5(x_3,\omega_3) \end{array}\right. \tag{26}$$

where $\overline{h}$ denotes the time average of $h(t)$, and, in order to avoid the introduction of new notations, the coordinates of the new vector y introduced in Lemma 3 are still denoted as x_3 and ω_3.

With the particular choice (8) for the functions g_i and h_i $(i=1,2)$, and setting:

$$L = \frac{a_1a_3}{2},\ M = -\frac{a_1a_2}{2},\ N = \frac{a_1^2}{2} \tag{27}$$

one easily verifies that:

$$\left\{\begin{array}{lcl} \overline{\dot{h}_1h_2} & = & L \\ \overline{h_1h_2} & = & \overline{\dot{h}_1\dot{h}_2} \ = \ -M \\ \overline{h_1^2} & = & \overline{(\dot{h}_1)^2} \ = \ N \end{array}\right. \tag{28}$$

Note that for any given triplet$(L,M,N) \in \mathbb{R}^2 \times \mathbb{R}_+^*$, there exists a corresponding triplet$(a_1,$
$a_2,a_3)$ of real numbers such that (27) is verified. Consequently, the numbers L, M and N may in turn be interpreted as independent design parameters.

Finally, we get from (8),(26) and (28):

$$\left\{\begin{array}{lcl} \dot{x}_3 & = & \frac{1}{2}\omega_3 - 2\alpha L x_3^3 + \omega_3 O_t^2(x_3,\omega_3) + O_t^5(x_3,\omega_3) \\ \dot{\omega}_3 & = & c_3[\omega_3^3(-2\alpha L - 4\beta M + N\alpha\beta) \\ & & \quad +\omega_3^2x_3(-6\alpha M + 4\beta L - 4N\beta^2) \\ & & \quad +\omega_3x_3^2(2\alpha L - 4\beta M - 8N\alpha\beta) \\ & & \quad +x_3^3(-4\alpha M - 4N\alpha^2)] + O_t^5(x_3,\omega_3) \end{array}\right. \tag{29}$$

From there, there only remains to show that for some values of α, β, L, M and N, the origin of the system (29) is asymptotically stable.

Lemma 4 *Consider the function* $V = 2Nc_3(\alpha^2 + 2\beta^2)x_3^4 + \frac{1}{2}\omega_3^2$ *. If*

$$L > 0, \quad M > 0, \quad N > 0, \quad \beta = \frac{L}{4N} \quad \text{and} \quad \alpha = \frac{2N\beta^2}{M} \quad , \tag{30}$$

then,

i) there exists $K > 0$ *such that, in a neighborhood of* $x_3 = \omega_3 = 0$*, we have*

$$\dot{V} \leq -K\,V^2 \tag{31}$$

where $\dot{V}$ *denotes the time-derivative of* V *along a solution of (29).*

ii) the origin of the system (29) is locally asymptotically stable (by application of "Lyapunov second method").

(Proof in the Appendix)

Since the conditions (30) are automatically satisfied when choosing the control parameters according to (9), we have proved Theorem 1.

4 Stabilization with torque controls

We now return to the problem of stabilizing the origin of the initial system (5) with u_1 and u_2, instead of ω_1 and ω_2, as control inputs. Note that since $u_1 + c_1\omega_2\omega_3 = \dot{\omega}_1$, and $u_2 + c_2\omega_1\omega_3 = \dot{\omega}_2$, the problem is basically equivalent to the one associated with the use of $\dot{\omega}_1$ and $\dot{\omega}_2$ as control inputs. This corresponds to the classical situation where integrators are added at the input level. The result, which is deduced from Theorem 1, is the following.

Theorem 2 *The smooth time-varying control law:*

$$\left\{ \begin{array}{lcll} u_1(X,\omega,t) & = & -c_1\omega_2\omega_3 + s_1(X,\omega,t) - k_3(\omega_1 - v_1(X,\omega_3,t)) & k_3 > 0 \\ u_2(X,\omega,t) & = & -c_2\omega_1\omega_3 + s_2(X,\omega,t) - k_4(\omega_2 - v_2(X,\omega_3,t)) & k_4 > 0 \end{array} \right. \tag{32}$$

with v_1 *and* v_2 *given by (7), (8) and (9) and,* s_1 *and* s_2 *their time derivatives along the trajectories of (5):*

$$s_i = \frac{1}{2}\frac{\partial v_i}{\partial X}(\,\omega \,+\, S(\omega)X \,+\, \langle\omega, X\rangle X\,) + \frac{\partial v_i}{\partial \omega_3}c_3\omega_1\omega_2 + \frac{\partial v_i}{\partial t} \qquad i = 1, 2.$$

locally asymptotically stabilizes the origin of (5).

The techniques used here are standard in the time-invariant case. The result of Theorem 2 can be proved by using the following lemma, which is a particular case of the small-gain theorem for nonlinear systems (see [23, Fact 4.1, Lemma 4.1]).

Lemma 5 *Consider the T-periodic system*

$$\left\{\begin{array}{lcl} \dot{x} & = & f(x,y,t)\ , \quad x \in \mathbb{R}^n \\ \dot{y} & = & g(y,t)\ , \quad y \in \mathbb{R}^m \end{array}\right. \tag{33}$$

where f and g are functions of class C^1. Assume that the origins of the two independent systems $\dot{x} = f(x,0,t)$ and $\dot{y} = g(y,t)$ are locally asymptotically stable. Then the origin $(x = 0, y = 0)$ of the coupled system (33) is locally asymptotically stable.

Setting $\tilde{\omega} = \omega - \eta$ with $\eta = (v_1, v_2, 0)$, the system (5) subjected to the controls (32) can be rewritten in the new coordinates $(X, \tilde{\omega})$:

$$\left\{\begin{array}{lcl} \dot{X} & = & \frac{1}{2}[\tilde{\omega} + \eta + S(\tilde{\omega}+\eta)X + \langle \tilde{\omega}+\eta, X\rangle X] \\ \dot{\tilde{\omega}}_3 & = & c_3\,(\tilde{\omega}_1+\eta_1)(\tilde{\omega}_2+\eta_2) \\ \dot{\tilde{\omega}}_1 & = & -k_3\tilde{\omega}_1 \\ \dot{\tilde{\omega}}_2 & = & -k_4\tilde{\omega}_2 \end{array}\right. \tag{34}$$

By applying lemma 5 to the closed-loop system (34), with $x = (X, \tilde{\omega}_3)$ and $y = (\tilde{\omega}_1, \tilde{\omega}_2)$, the local stability of the origin of this system stems from the local stability, previously established in Theorem 1, of the reduced system obtained by setting $y = 0$.

5 Estimation of asymptotical convergence rates

Let us first study the convergence rates associated with the closed-loop reduced system.
In view of (31), the Lyapunov function $V(t)$ of Lemma 4 converges to zero at least as fast as t^{-1} (modulo some multiplicative constant). This in turn implies, using the fact that the change of coordinates $\phi : (x,t) \longmapsto (y,t)$ of Lemma 3 is such that $|y/x|$ uniformly tends to one as $|x|$ tends to zero, that $x_3(t)$ and $\omega_3(t)$ tend to zero at least as fast as $t^{-\frac{1}{4}}$ and $t^{-\frac{1}{2}}$ respectively. We also know (see the proof of Lemma 2 in the Appendix) that, the solutions of the system (12) are such that $|z_1(t)| = |O_t^3(x_3(t),\omega_3(t))|$ and $|z_2(t)| = |O_t^4(x_3(t),\omega_3(t))|$. This in turn implies, in view of (10), that $|x_1(t)| \leq |g_1(x_3(t),\omega_3(t))h_1(t)| + |O_t^3(x_3(t),\omega_3(t))|$ and $|x_2(t)| \leq |g_2(x_3(t),\omega_3(t))h_2(t)| + |O_t^4(x_3(t),\omega_3(t))|$. Since $g_1(x_3,\omega_3)$ and $g_2(x_3,\omega_3)$

are homogeneous polynomials of degree one and two respectively, we can conclude now, that $x_1(t)$ and $x_2(t)$ converge to zero at least as fast as $t^{-\frac{1}{4}}$ and $t^{-\frac{1}{2}}$ respectively. Finally, one deduces from (7) that the controls $v_1(t) \equiv \omega_1(t)$ and $v_2(t) \equiv \omega_2(t)$ converge to zero at least as fast as $t^{-\frac{1}{4}}$ and $t^{-\frac{1}{2}}$ respectively.

It can be shown, either by applying center manifold results to the system (34) or by close inspection of the proof of the small gain theorem used in Lemma 5, that the above estimation of convergence rates obtained for the closed-loop reduced system, remains valid when the torques controls (32) are applied to the spacecraft.

The action of the controls (32) on the system (5) has been simulated with the following choice of parameters:

$$k_1 = k_2 = 1; \quad a_1 = 3; \quad a_2 = -3; \quad a_3 = 4.$$

The **figures 1-4** show the time evolution of x_1, x_2, x_3, and ω_3 when the initial conditions are:

$$(x_1(0), x_2(0), x_3(0), \omega_1(0), \omega_2(0), \omega_3(0))^T = (0.1, -0.1, 0.05, -0.1, 0.2, 0.2)^T.$$

We have verified that the convergence rates observed from this simulation are slightly better than $t^{-\frac{1}{2}}$ for x_2 and ω_3, and $t^{-\frac{1}{4}}$ for x_1 and x_3. This is in accordance with the above analysis which provides lower bounds only.

6 Conclusion

In this paper, a smooth time-periodic control feedback law which asymptotically stabilizes the attitude of a rigid spacecraft, in the case where only two independent control torque actions are available, each being exerted about a principal axis of inertia, has been derived. Existence of such a control had previously been established in [12, 7], without an explicit and analytical expression of the control being given. The interest of this time-varying solution is that it circumvents the theoretical impossibility of achieving attitude stabilization by means of a continuous time-invariant feedback.

It is not difficult to extend this solution to the case where only one of the torque actions is exerted about a principal axis of inertia. If the spacecraft possesses an axis of symetry, it is possible, except for particular configurations of the two actuators, to fall back upon this latter case by considering an adequate linear combination of the control actions. Nevertheless, the extension of the proposed design method to the general case, when none of the control torques is exerted about a principal axis of inertia, is not straightforward and thus remains to be treated in connection with other robustness issues.

Global convergence results and obtention of a faster rate of convergence, which could be of practical interest, will also motivate future studies.

Appendix : Proof of Lemmas 1-4

Proof of Lemma 1

Let us rewrite (6) as :

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & \frac{1}{2}v_1 + \Gamma_1(X,\omega_3,v_1,v_2) \\ \dot{x}_2 & = & \frac{1}{2}(v_2 + x_3v_1 - x_1\omega_3) + \Gamma_2(X,\omega_3,v_1,v_2) \\ \dot{x}_3 & = & \frac{1}{2}\omega_3 + \Gamma_3(X,\omega_3,v_1,v_2) \\ \dot{\omega}_3 & = & c_3v_1v_2 \end{array}\right. \tag{35}$$

with

$$\left\{\begin{array}{rcl} \Gamma_1 & = & \frac{1}{2}[\,-x_3v_2 + x_2\omega_3 + x_1(v_1x_1 + v_2x_2 + \omega_3x_3))\\ \Gamma_2 & = & \frac{1}{2}x_2(v_1x_1 + v_2x_2 + \omega_3x_3) \\ \Gamma_3 & = & \frac{1}{2}(-x_2v_1 + x_1v_2 + x_3(v_1x_1 + v_2x_2 + \omega_3x_3))\ . \end{array}\right. \tag{36}$$

System (35) may then be rewritten in the coordinates $(z_1, z_2, x_3, \omega_3)$ as :

$$\left\{\begin{array}{rcl} \dot{z}_1 & = & -k_1z_1 - h_1\frac{\partial g_1}{\partial x_3}\Lambda_3 - h_1\frac{\partial g_1}{\partial \omega_3}c_3v_1v_2 + \Lambda_1 \\ \dot{z}_2 & = & -k_2z_2 - h_2\frac{\partial g_2}{\partial x_3}\Lambda_3 - h_2\frac{\partial g_2}{\partial \omega_3}c_3v_1v_2 + \Lambda_2 \\ \dot{x}_3 & = & \frac{1}{2}\omega_3 + \Lambda_3 \\ \dot{\omega}_3 & = & c_3v_1v_2 \end{array}\right. \tag{37}$$

with

$$\Lambda_i(z_1,z_2,x_3,\omega_3,t) \ = \ \Gamma_i(z_1+g_1h_1, z_2+g_2h_2, x_3, \omega_3, v(z_1,z_2,x_3,\omega_3,t)), \quad i=1,2,3 \tag{38}$$

where $v(z_1,z_2,x_3,\omega_3,t)$ stands for $(v_1(z_1,x_3,\omega_3,t), v_2(z_1,z_2,x_3,\omega_3,t))$, as given by (11).

Define

$$\begin{array}{rcl} l_1(z,x_3,\omega_3,t) & = & -\,h_1\frac{\partial g_1}{\partial x_3}\Lambda_3 - h_1\frac{\partial g_1}{\partial \omega_3}c_3v_1v_2 + \Lambda_1\ , \\ l_2(z,x_3,\omega_3,t) & = & -h_2\frac{\partial g_2}{\partial x_3}\Lambda_3 - h_2\frac{\partial g_2}{\partial \omega_3}c_3v_1v_2 + \Lambda_2\ , \\ f_1(z,x_3,\omega_3,t) & = & \Lambda_3\ , \\ f_2(z,x_3,\omega_3,t) & = & c_3\,v_1v_2\ , \end{array} \tag{39}$$

so that (37) can be rewritten as (12).
The relations (13) are easily obtained from (39)-(38)-(36).
From (11), and using the fact that g_1 and g_2 are homogeneous polynomials of degree 1 and 2 and h_1 and h_2 are periodic functions of time, we get

$$\left\{ \begin{array}{lcl} v_1(0, x_3, \omega_3, t) & = & O_t(x_3, \omega_3) \\ v_2(0, 0, x_3, \omega_3, t) & = & O_t^2(x_3, \omega_3) \end{array} \right. \tag{40}$$

so that, from (38) and (36)

$$\left\{ \begin{array}{lcl} \Lambda_1(0, 0, x_3, \omega_3, t) & = & O_t^3(x_3, \omega_3) \\ \Lambda_3(0, 0, x_3, \omega_3, t) & = & O_t^3(x_3, \omega_3) \\ v_1(0, x_3, \omega_3, t) v_2(0, 0, x_3, \omega_3, t) & = & O_t^3(x_3, \omega_3) \ . \end{array} \right. \tag{41}$$

The relation (14) is then easily obtained from (39) and (41). The proof of (15) is similar. □

Proof of Lemma 2

The proof of this lemma involves an extension of the classical results of center manifold theory to time-periodic systems.

Proposition 1 (Existence of a center manifold) *Consider the system:*

$$\left\{ \begin{array}{l} \dot{z} = Bz + l(z, x, t) \\ \dot{x} = Ax + f(z, x, t) \end{array} \right. \tag{42}$$

with $z \in \mathbb{R}^n, x \in \mathbb{R}^m$, B a matrix with eigenvalues having negative real parts, A a matrix with eigenvalues having zero real parts, l and f T-periodic functions of class C^2 vanishing at $(0, 0, t)$ and whose first derivatives also vanish at these points. Then, there exists $\delta > 0$ and $\pi(x, t) : \mathbb{R}^m \times \mathbb{R} \longrightarrow \mathbb{R}^n$ a T-periodic function of class C^2 such that the set $\{ (\pi(x, t), x) \ |x| < \delta, \ t \in \mathbb{R} \}$ defines a local center manifold for (42), i.e.

- *i) π vanishes at $(0, t)$ and its first derivatives also vanish at these points,*
- *ii) solutions of (42) starting in the set $\{ (\pi(x, t), x) \ |x| < \delta, \ t \in \mathbb{R} \}$ remain in this set as long as $|x(t)| < \delta$, i.e.*

$$\frac{\partial \pi}{\partial t} + \frac{\partial \pi}{\partial x}(x, t) \left(Ax + f(\pi(x, t), x, t) \right) \ = \ B\pi(x, t) + l(\pi(x, t), x, t) \quad \text{for } |x| < \delta. \tag{43}$$

This result is a particular case of [11, Th 1 and section 5].

Proposition 2 (Reduction principle) *Let π be a function satisfying the conditions i) and ii) of Proposition 1. If the origin of the corresponding "reduced system" :*

$$\dot{x} = Ax + f(\pi(x,t), x, t) \tag{44}$$

is locally asymptotically stable, then so is the origin of (42). Moreover, in this case, if (z_0, x_0) is small enough, and $(z(t), x(t))$ is the solution of (42) with initial values (z_0, x_0) at $t = t_0$, then there exists a solution $u(t)$ of (44) such that:

$$\left\{ \begin{array}{l} x(t) - u(t) = O(e^{-\gamma t}) \\ z(t) - \pi(u(t), t) = O(e^{-\gamma t}) \end{array} \right. \tag{45}$$

where γ depends only on the eigenvalue of B with largest real part.

Proposition 3 (Approximation) *Let π be a function satisfying the conditions i) and ii) of Proposition 1, and $\Phi : \mathbb{R}^m \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be a T-periodic function of class C^1 such that:*

i) Φ vanishes at $(0, t)$ and its first derivatives also vanish at these pointsl,

ii) $\frac{\partial \Phi}{\partial x}(x,t)\,[Ax + f(\Phi(x,t), x, t)] - B\Phi - l(\Phi(x,t), x, t) + \frac{\partial \Phi}{\partial t} = O_t^q(x)$.

Then, $\pi(x,t) - \Phi(x,t) = O_t^q(x)$.

For the proof of these two propositions, we refer to [5]. The main reason why the proofs still hold in the time-periodic case is that the relations (2.3.5) and (2.4.10) in [5] are not modified by the introduction of the time variable when l and f are time-periodic.

We now proceed with the proof of Lemma 2. From Proposition 1, there exists for the system (16) a center manifold defined by $z_1 = \pi_1(x,t), \ldots, z_n = \pi_n(x,t)$. If we prove :

$$\left\{ \begin{array}{ll} \pi_1(x,t) & = O_t^{q_1}(x) \ , \\ \quad \vdots & \\ \pi_i(x,t) & = O_t^{q_i}(x) \ , \\ \quad \vdots & \\ \pi_n(x,t) & = O_t^{q_n}(x) \ , \end{array} \right. \tag{46}$$

then, Lemma 2 directly follows from Proposition 2. Let us prove (46) by induction.

First, it follows from (17) and (18) that $l(0,x,t) = O_t^{q_1}(x)$ and, by application of Proposition 3 with $\Phi = 0$, we deduce that $\pi_i(x,t)$, for $i = 1,\ldots,n$, is at least a $O_t^{q_1}$ function. In particular, $\pi_1(x,t) = O_t^{q_1}(x,t)$.Let us now assume that $\pi_1(x,t) = O_t^{q_1}(x),\ldots,\pi_k(x,t) = O_t^{q_k}(x)$ and prove that $\pi_{k+1}(x,t) = O_t^{q_{k+1}}(x)$. To this purpose, we consider the following reduced-order system:

$$\begin{array}{rcl} \dot{z}_{k+1} & = & b_{k+1,k+1}\, z_{k+1} + \ldots + b_{k+1,n}\, z_n + l_{k+1}(\pi_1(x,t),..,\pi_k(x,t), z_{k+1},.., z_n, x, t) \\ \vdots & & \qquad\qquad \ddots \\ \dot{z}_n & = & \qquad\qquad\qquad b_{n,n}\, z_n + l_n(\pi_1(x,t),\ldots,\pi_k(x,t), z_{k+1},\ldots,z_n,x,t) \\ \dot{x} & = & Ax + f(\pi_1(x,t),\ldots,\pi_k(x,t), z_{k+1},\ldots,z_n,x,t) \end{array} \tag{47}$$

where the $b_{i,j}$'s are the entries of B (recall that it is upper-triangular and $b_{i,i} < 0$). Since $x \mapsto (\pi_{k+1}(x,t),\ldots,\pi_n(x,t))$ satisfies the conditions i) and ii) of Proposition 1, and since $\Phi = 0$ satisfies the conditions i) and ii) of Proposition 3 (from (17), (18) and the induction hypothesis), Proposition 3 applies with z replaced by $(z_{k+1},\ldots,z_n)$, π replaced by $(\pi_{k+1},\ldots,\pi_n)$, $\Phi = 0$ and $q = q_{k+1}$. Hence, $\pi_j(x,t)$, for $j = k+1,\ldots,n$, is a $O_t^{q_{k+1}}$ function, and in particular $\pi_{k+1}(x,t) = O_t^{q_{k+1}}(x)$. □

Proof of Lemma 3

The proof is given for $n = 2$ and $k = 3$ (the spacecraft's case). The general case is just a simple extension.

We assume, without loss of generality, that the components of the vector $\sigma(x)$ in (24) form a basis for the homogeneous polynomials of degree three. We take $\sigma(x) = (x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3)$.

The change of coordinates evoked in the lemma is of the same type as the one used in [22], or originally in [10, pp 168], and is of the form

$$x = y + H(t)\sigma(y) \tag{48}$$

with H a 2×4 matrix of T-periodic functions which are specified further in the proof. This change of coordinates transforms (24) into:

$$\begin{array}{rcl} \dot{y} & = & [Id + H(t)\sigma'(y)]^{-1}[Ay + AH(t)\sigma(y) + D(t)\sigma(y + H(t)\sigma(y)) \\ & & -\dot{H}(t)\sigma(y) + O_t^5(y)] \end{array} \tag{49}$$

(where $\sigma'(y)$ is the Jacobian matrix of $\sigma(y)$). Note that $[Id + H(t)\sigma'(y)]^{-1}$ is defined in a neighborhood of $y = 0$ for all t.

Using a classical formula on matrix inverses, and the fact that $\sigma(y) = O^3(y)$, we easily obtain from (49):

$$\begin{array}{rcl} \dot{y} & = & [Id - H(t)\sigma'(y) + O^4(y)][Ay + AH(t)\sigma(y) + D(t)\sigma(y) - \dot{H}(t)\sigma(y) \\ & & +O_t^5(y)] \\ & = & Ay + \overline{D}\sigma(y) + Z(t,y) + O_t^5(y) \end{array}$$

with

$$\begin{array}{lcl} Z(t,y) & = & \tilde{D}(t)\sigma(y) + AH(t)\sigma(y) - \dot{H}(t)\sigma(y) - H(t)\sigma'(y)Ay, \\ \overline{D} & = & (\frac{1}{T}\int_0^T D(t)\,dt), \\ \tilde{D}(t) & = & D(t) - \overline{D}. \end{array} \tag{50}$$

Let us find an appropriate choice of H which makes $Z(t,y)$ vanish. Since the entries of $\sigma'(y)Ay$ are homogeneous polynomials of degree 3, there exists a (unique) 4×4 matrix C such that $\sigma'(y)Ay = C\sigma(y)$. Since A is strictly upper triangular, one easily verifies that C is strictly upper triangular too. Now, from the above expression of Z, and using $\sigma'(y)Ay = C\sigma(y)$, $Z(t,y)$ is identically zero if and only if $H(t)$ is a solution of

$$\dot{H} = \widetilde{D}(t) + AH - HC \ \ , \tag{51}$$

i.e. if the functions $h_{ij}(t)$, $i = 1,2$, $j = 1,\ldots,4$ satisfy for all t:

$$\left\{ \begin{array}{lcl} \dot{h}_{21}(t) & = & \widetilde{d}_{21}(t) \\ \dot{h}_{22}(t) & = & \widetilde{d}_{22}(t) - c_{12}h_{21}(t) \\ & \vdots & \\ \dot{h}_{11}(t) & = & \widetilde{d}_{11}(t) + a_{12}h_{21}(t) \\ \dot{h}_{12}(t) & = & \widetilde{d}_{12}(t) + a_{12}h_{22}(t) - c_{12}h_{11}(t) \\ & \vdots & \\ \dot{h}_{14}(t) & = & \widetilde{d}_{14}(t) + a_{12}h_{24} - c_{14}h_{11}(t) - c_{24}h_{12}(t) - c_{34}h_{13}(t) \end{array} \right. \tag{52}$$

The triangular structure of this system plus the fact that $\widetilde{d}_{i,j}(t)$ is time-periodic and has zero-mean value allows one to build recursively a solution which shares the same properties: start with $h_{21}(t) = \displaystyle\int_0^t \tilde{d}_{21}(s)\,ds - \frac{1}{T}\int_0^T (\int_0^u \tilde{d}_{21}(s)\,ds)\,du$, and continue with $h_{22}(t),\ldots,h_{24}(t),h_{11}(t),\ldots,h_{14}(t)$. □

Proof of Lemma 4

From (29), we have :

$$\dot{V}(x_3,\omega_3) \ \ = \ \ c_3\omega_3^4(-2\alpha L - 4\beta M + N\alpha\beta)$$

$$\begin{aligned}
&+c_3\omega_3^3x_3(-6\alpha M+4\beta L-4N\beta^2)\\
&+c_3\omega_3^2x_3^2(2\alpha L-4\beta M-8N\alpha\beta)\\
&+\omega_3x_3^3(c_3(-4\alpha M-4N\alpha^2)+4Nc_3(\alpha^2+2\beta^2))\\
&-16NLc_3\alpha(\alpha^2+2\beta^2)x_3^6\\
&+\omega_3O_t^5(x_3,\omega_3)+O_t^8(x_3,\omega_3)
\end{aligned}$$

Using (30), we get after some calculations

$$\dot{V}(x_3,\omega_3)\leq -c_3\,W(x_3,\omega_3)+\omega_3O_t^5(x_3,\omega_3)+O_t^8(x_3,\omega_3) \tag{53}$$

with:

$$W(x_3,\omega_3)=\left[\frac{LM}{N}\omega_3^4+\frac{LM}{N}\omega_3^2x_3^2+\frac{32N^2\beta^2L(\alpha^2+2\beta^2)}{M}x_3^6\right] \tag{54}$$

Obviously:

$$W(x_3,\omega_3)\geq\frac{2K}{c_3}(V(x_3,\omega_3))^2\quad\text{when } |(x_3,\omega_3)|<\epsilon, \tag{55}$$

for some positive numbers ϵ and K.
Moreover, $|O_t^8(x_3,\omega_3)|/W(x_3,\omega_3)$ uniformly converges to zero when $|(x_3,\omega_3)|$ tends to zero, and so does $|\omega_3O_t^5(x_3,\omega_3)|/W(x_3,\omega_3)$ since:

$$\begin{aligned}
|\omega_3O_t^5(x_3,\omega_3)| &\leq |(x_3,\omega_3)|.|\omega_3|.|(x_3,\omega_3)|.|O_t^3(x_3,\omega_3)|\\
&\leq \frac{1}{2}|(x_3,\omega_3)|(\omega_3^2(x_3^2+\omega_3^2)+O_t^6(x_3,\omega_3))\\
&\leq \frac{1}{2}|(x_3,\omega_3)|(\omega_3^4+x_3^2\omega_3^2+O_t^6(x_3,\omega_3))
\end{aligned}$$

Hence, there exists a positive number ϵ_1 $(\leq\epsilon)$ such that:

$$|\omega_3\,O_t^5(x_3,\omega_3)|+|O_t^8(x_3,\omega_3)|\leq\frac{c_3}{2}W(x_3,\omega_3)\quad\text{when } |(x_3,\omega_3)|<\epsilon_1 \tag{56}$$

The relation (31) of Lemma 4 follows from (53), (55) and (56). □

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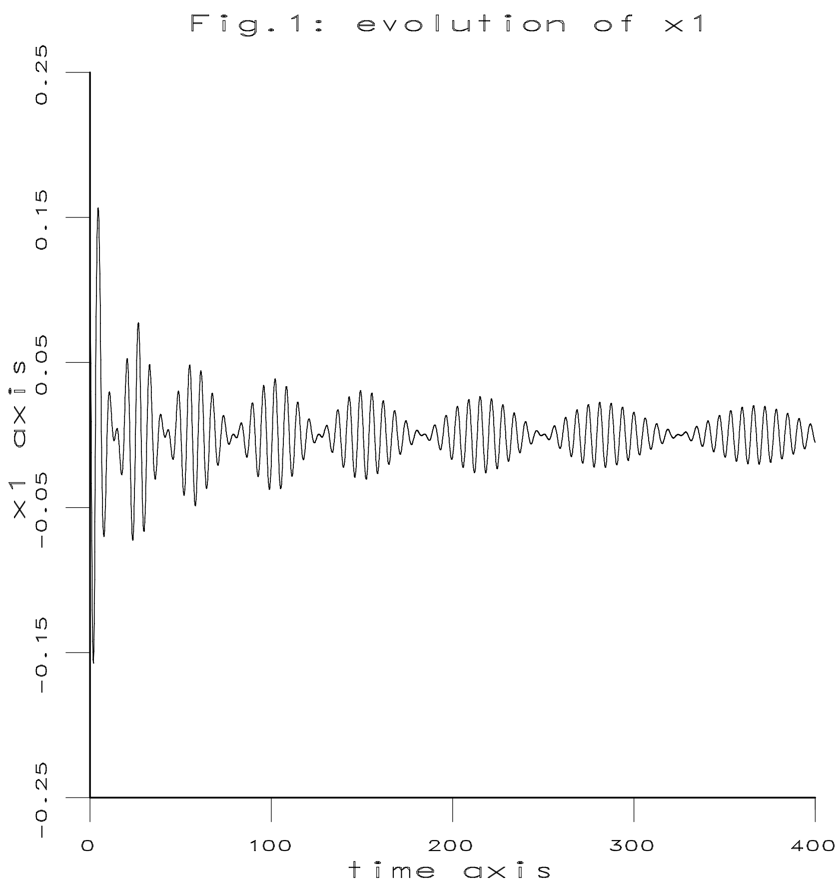
Fig.1: evolution of x1
x1 axis
0.25
0.15
0.05
-0.05
-0.15
-0.25
0
100
200
300
400
time axis

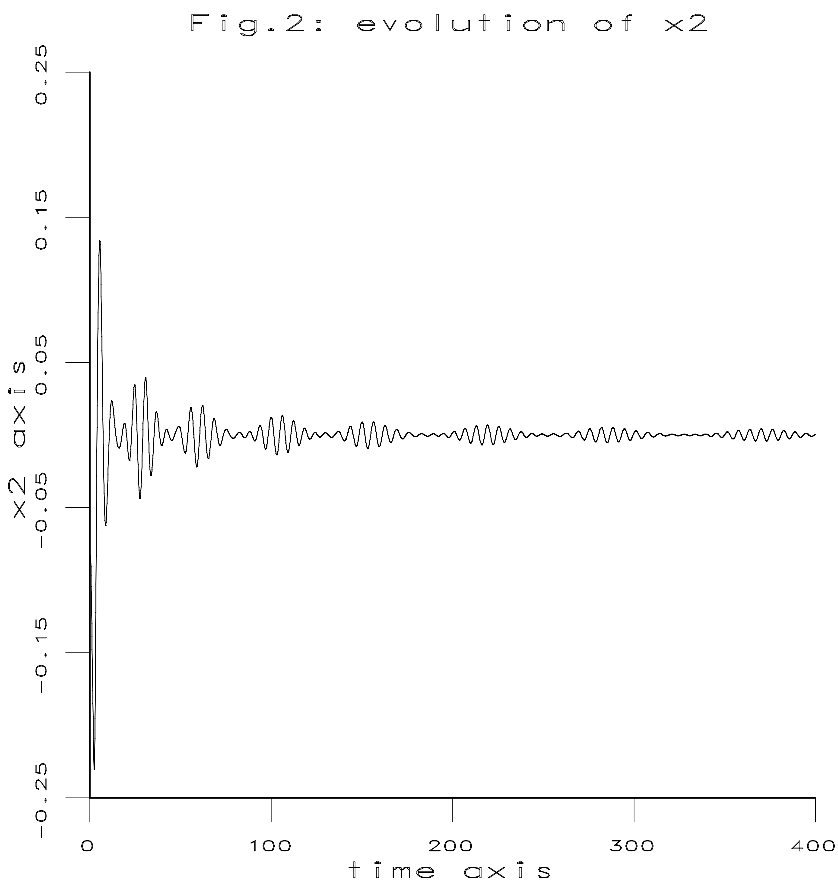
Fig.2: evolution of x2
x2 axis
0.25
0.15
0.05
-0.05
-0.15
-0.25
0
100
200
300
400
time axis

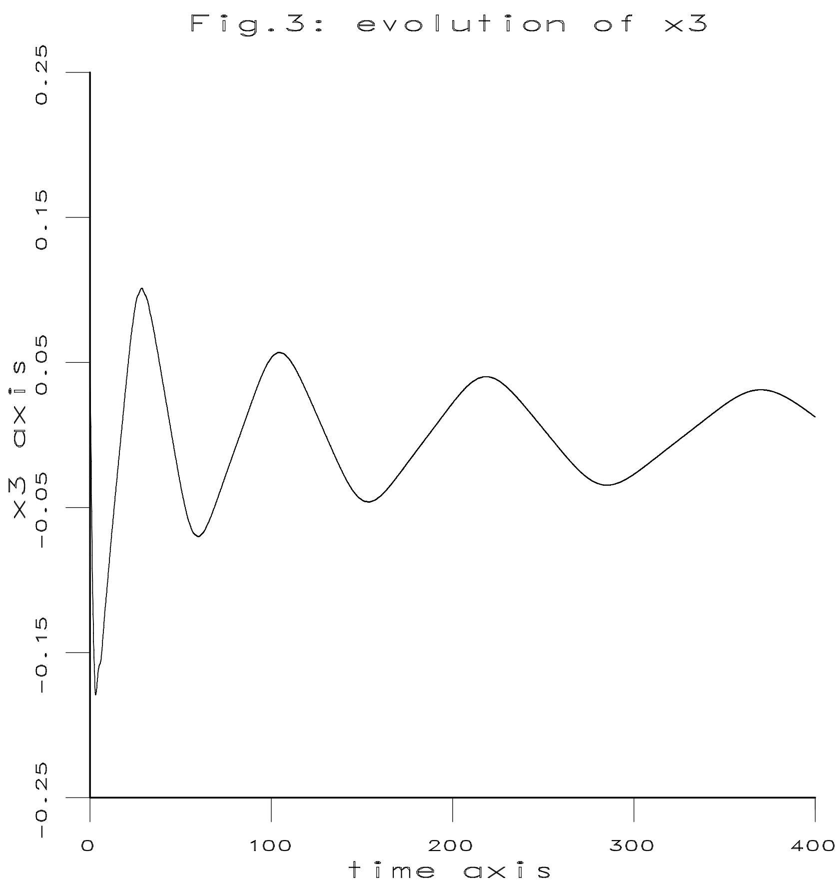
Fig.3: evolution of x3
x3 axis
time axis
0.25
0.15
0.05
-0.05
-0.15
-0.25
0
100
200
300
400

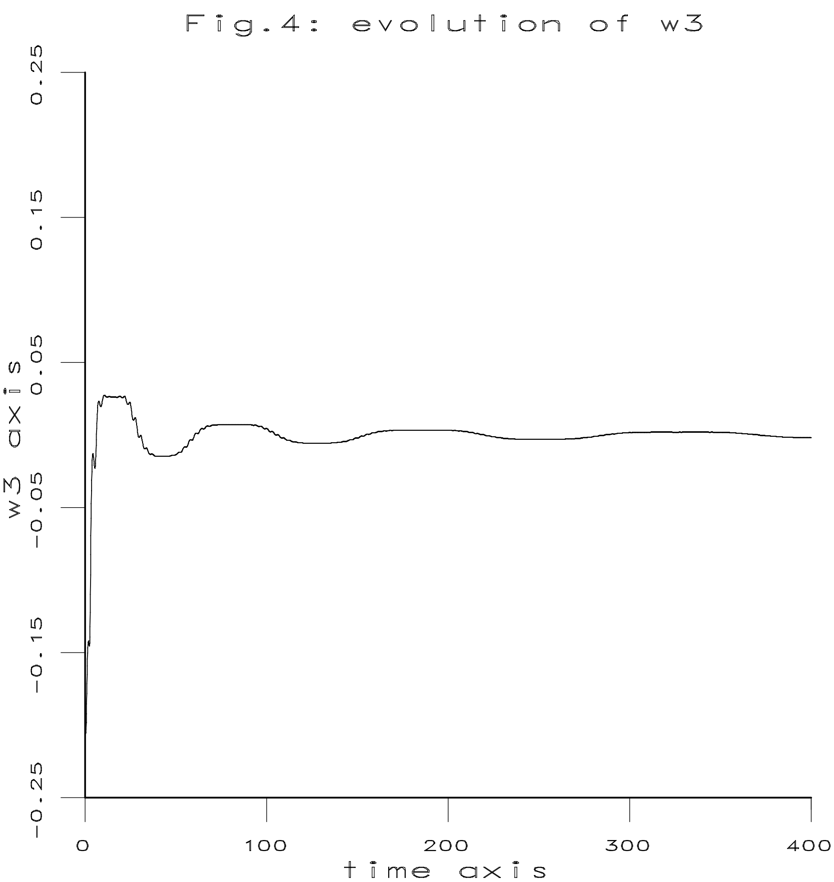
Fig.4: evolution of w3
w3 axis
0.25
0.15
0.05
-0.05
-0.15
-0.25
0
100
200
300
400
time axis

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